ON BESSEL-RIESZ OPERATORS

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ABSTRACT: We consider a class of conv olution operator denoted $W^{\alpha}\varphi$ obtained by convolution with a generalized function expressible in terms of the Bessel function on first kind J_{γ} with argument the distribution $(P \pm i0)$. We study some elementary properties of the operator $W^{\alpha}\varphi$ like the semigroup property $W^{\alpha}W^{\beta}\varphi = W^{\alpha+\beta}\varphi$; and $(\square + m^2)\,W^{\alpha}\varphi = W^{\alpha-2}$ for $\alpha > 2$ where $(\square + m^2)$ is the Klein-Gordon ultrahyperbolic operator. Moreover we prove that the operator $W^{\alpha}\varphi$ may be consider as a negative power of the Klein-Gordon operator

Key words: Bessel-Riesz potentials, fractional derivative, hypersingular integral

I. INTRODUCTION

This article deals with certain kind of potential operator defined as convolution with the generalized function $W_{\alpha}(P \pm i0, m, n)$ depending on a complex parameter α and a real non negative one *m*.

The definitory formulae and several properties of the family $\{W_{\alpha}(P \pm i0, m, n)\}_{\alpha}; \alpha \in C$ have been introduced and studied by Trione (see [14]) specially the important followings two:

a) $W_{\alpha} * W_{\beta} = W_{\alpha+\beta}$, α and β complex numbers, and

b) W_{-2k} is a fundamental solution of the *k*-times iterated Klein-Gordon operator

Writing $W_{\alpha}(P \pm i0, m, n)$ as an infinite linear combination of the ultrahyperbolic Riesz kernel of different orders $R_{\alpha}(P \pm i0)$ which is a causal (anticausal) elementary solution of the ultrahyperbolic differential operator and taking into account its Fourier transform it is possible to evaluate the Fourier transform of the kernel $W_{\alpha}(P \pm i0, m, n)$.

We prove the composition formula $W^{\alpha} * W^{\beta}\varphi = W^{\alpha+\beta}\varphi$ for a sufficiently good function. The proof of this result is based on the composition formulae presented by Trione in [14], but we also present a different way.

Other simple property studied is the one that establish the relationship between the ultrahyperbolic Klein-Gordon operator and the W^{α} Bessel-Riesz operator.

Finally we obtain an expression that will be consider a fractional power of the Klein-Gordon operator.

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II. PRELIMINARY DEFINITIONS AND RESULTS

Let $t=(t_1,t_2,\cdots,t_n)$ be a point of the n-dimensional space R^n. Let $P=P(t)$ be the quadratic non degenerate form in n variables

$$P=P(t)=t_1^2+\cdots+t_p^2-t_{p+1}^2-\cdots-t_{p+q}^2 \qquad \text{(II.1)}$$

where $p+q=n$

Gelfand (cf.[4]) introduced the $(P\pm i0)^\lambda$ distributions as the following limit

$$(P\pm i0)^\lambda=\lim_{\varepsilon\to 0}\left(P\pm i\varepsilon|t|^2\right)^\lambda \qquad \text{(II.2)}$$

where $\varepsilon>0$, $|t|^2=t_1^2+\cdots+t_n^2$ and λ is a complex number.

Frequently we use an equivalent expression given by

$$(P\pm i0)^\lambda=P^\lambda+e^{\pm i\pi\lambda}P^\lambda \qquad \text{(II.3)}$$

where the generalized functions P_+^λ and P_-^λ are defined by

$$P_+^\lambda=\begin{cases}P^\lambda & \text{if } P\geq 0\\ 0 & \text{if } P<0\end{cases}$$

and

$$P_-^\lambda=\begin{cases}0 & \text{if } P>0\\ |P|^\lambda & \text{if } P\leq 0\end{cases}$$

It is well known (cf.[4]) the Fourier transform of generalized functions associated with a quadratic form and in the particular case of $(P\pm i0)^\lambda$ it results

$$F\left[(P\pm i0)^\lambda\right]=\frac{1}{(2\pi)^{\frac{n}{2}}}\int_{R^n}e^{-i(t,y)}(P\pm i0)^\lambda\,dt=C_{(\lambda,q)}\cdot(Q\mp i0)^{-\lambda-\frac{n}{2}} \qquad \text{(II.4)}$$

where

$$C_{(\lambda,q)}=\frac{e^{\pm\frac{n}{2}q}\,2^{2\lambda+n}\pi^{\frac{n}{2}}\Gamma\left(\lambda+\frac{n}{2}\right)}{(2\pi)^{\frac{n}{2}}\,\Gamma(-\lambda)} \qquad \text{(II.5)}$$

and $Q=Q(y)=y_1^2+\cdots+y_p^2-y_{p+1}^2-\cdots-y_{p+q}^2$; $\langle t,y\rangle=\sum_{i=1}^n t_iy_i$. (II.6)

where m is a positive real number; $J_\gamma(z)$ the Bessel function of first kind

$$J_\gamma(z)=\sum_{p=0}^{\infty}\frac{(-1)^P\left(\frac{z}{2}\right)^{\gamma+2p}}{p!\,\Gamma(p+\gamma+1)}$$

and $\Gamma(z)$ is the gamma function

We start by observing that the family $\{W_\alpha(P \pm i0, m, n)\}_\alpha ; \alpha \in C$ is a certain kind of generalization of the family of retarded functions supported in the light cone introduced by Marcel Riesz (cf.[7]) and by L. Schwartz (cf. [11]) and studied by Trione (cf. [5]) defined by

$$W_\alpha(u,m) = \begin{cases} \dfrac{\left(m^{-2}u\right)^{\frac{\alpha+n}{4}}}{\pi^{\frac{n+2}{2}} 2^{\frac{\alpha+n-2}{2}} \Gamma\left(\frac{\alpha}{2}\right)} J_{\frac{\alpha-n}{2}}\left[\left(m^2 u\right)^{\frac{1}{2}}\right] & \text{if } t \in \Gamma_+ \\ 0 & \text{if } t \notin \Gamma_+ \end{cases} \qquad \text{(II.7)}$$

where $u = t_1^2 - t_2^2 - \cdots - t_n^2$ and Γ_+ is the cone

$$\Gamma_+ = \left\{ t \in R^n : t_1 > 0, u > 0 \right\}$$

$W_\alpha(u,m)$ that is an ordinary function if $\operatorname{Re}\alpha \geq n$ is a distributional entire function on α (cf [5]).

If in (II.7) we replace $J_{\frac{\alpha-n}{2}}$ by its Taylor series, when $m = 0$ we obtain the ultrahyperbolic kernel due by Nozali (cf [6]), given by

$$\Phi_\alpha = \frac{\Gamma_+^{\alpha-n}}{C_n(\alpha)} \qquad \text{(II.8)}$$

where

$$\Gamma_+^{\alpha-n} = \left(t_1^2 + \cdots + t_p^2 - t_{p+1}^2 - \cdots - t_{p+q}^2\right)^{\frac{\alpha-n}{2}} ;\ t_1 > 0;\ p + q = n$$

and

$$C_n(\alpha) = \frac{\pi^{\frac{n-1}{2}} \Gamma\left(\frac{2-\alpha-n}{2}\right) \Gamma(\alpha)}{\Gamma\left(\frac{2+\alpha-p}{2}\right) \Gamma\left(\frac{p-\alpha}{2}\right)} \qquad \text{(II.9)}$$

By putting $p = 1$ in (II.8) and (II.9) we obtain inmediately

$$R_\alpha(u) = \begin{cases} \dfrac{u^{\frac{\alpha-n}{2}}}{H_m(\alpha)} & \text{if } t \in \Gamma_+ \\ 0 & \text{if } t \notin \Gamma_+ \end{cases} \qquad \text{(II.10)}$$

where

$$H_m(\alpha) = 2^{\alpha-1} \pi^{-1+\frac{n}{2}} \Gamma\left(\frac{\alpha}{2}\right) \Gamma\left(\frac{\alpha + 2 - n}{2}\right)$$

$R_\alpha(u)$ is the hyperbolic kernel introduced by Riesz.

By putting $n = 1$ in $R_\alpha(u)$, and taking into account the Legendre's duplication formula of $\Gamma(z)$:

$$\Gamma(2z) = 2^{2z-1} \pi^{\frac{1}{2}} \Gamma(z) \Gamma\left(z + \frac{1}{2}\right)$$

we get

$$I_\alpha = \begin{cases} \dfrac{t^{\frac{\alpha-1}{2}}}{\Gamma(\alpha)} & \text{if } t > 0 \\ 0 & \text{if } t < 0 \end{cases} \tag{II.11}$$

Or, equivalently $I_\alpha = \dfrac{t_+^{\frac{\alpha-1}{2}}}{\Gamma(\alpha)}$, where $t_+^{\frac{\alpha-1}{2}}$ is the distribution defined by

$$t_+^\lambda = \begin{cases} t^\lambda & \text{if } t > 0 \\ 0 & \text{if } t < 0 \end{cases} \tag{II.12}$$

(cf. [4]). I_α is precisely the singular kernel of Riemann-Liouville studied by Riesz (cf. [7]) and also by Trione [12].

Definition 1. Let φ be a sufficiently good function, we introduce the convolution type operator $W^\alpha\varphi$

$$W^\alpha\varphi = W_\alpha(P \pm i0, m, n) * \varphi \tag{II.13}$$

which is defined in Fourier transform by the following equality

$$\Im[W^\alpha\varphi] = \Im[W_\alpha] \cdot \Im[\varphi] \tag{II.14}$$

Because the function $W_\alpha(P \pm i0, m, n)$ is expressed in terms of Bessel functions of first kind and that when $m = 0$ it reduces at the Marcel Riesz ultrahyperbolic kernel $R_\alpha(P \pm i0)$ (cf[14]) is why the operator (II.13) is called the Bessel-Riesz potential.

From the definitory formula of $J_\gamma(z)$, and putting by definition according Trione (cf. [14])

$$\binom{-\frac{\alpha}{2}}{\lambda}\Gamma\left(\frac{\alpha}{2}\right) = (-1)^\gamma \frac{1}{\gamma!}\Gamma\left(\frac{\alpha}{2}+\gamma\right) \tag{II.15}$$

$$\text{and } H_n(\alpha+2\gamma) = \frac{2^{\alpha+2\gamma}\pi^{\frac{n}{2}}\Gamma\left(\frac{\alpha+2\gamma}{2}\right)}{\Gamma\left(\frac{n-\alpha-2\gamma}{2}\right)}$$

it results that the generalized function $W_\alpha(P \pm i0, m, n)$ may be expressed as an infinite linear combination of the ultrahyperbolic causal (anticausal) Riesz kernel

$$W_\alpha(P \pm i0, m, n) = \sum_{\gamma=0}^{\infty}\binom{-\frac{\alpha}{2}}{\gamma} m^{2\gamma}\frac{(P \pm i0)^{\frac{\alpha-n+2\gamma}{2}}}{H_n(\alpha+2\gamma)} \tag{II.16}$$

This formula allow us to write the Fourier transform of W_α as

$$\Im\left[W^{\alpha}\varphi\right]=\sum_{\gamma=0}^{\infty}\binom{-\frac{\alpha}{2}}{\gamma}m^{2\gamma}\left(Q\mp i0\right)^{-\frac{\alpha+2\gamma}{2}}\Im[\varphi] \qquad \text{(II.17)}$$

Taking into account (II.13) and (II.16) the operator $W^{\alpha}\varphi$ has the form

$$W^{\alpha}\varphi=\sum_{\gamma=0}^{\infty}\binom{-\frac{\alpha}{2}}{\gamma}m^{2\gamma}\left[\int_{K_+}P^{\frac{\alpha-n+2\gamma}{2}}\varphi(x-t)dt+e^{\frac{i\pi(\alpha-n+2\gamma)}{2}}\int_{K_-}|P|^{\frac{\alpha-n+2\gamma}{2}}\varphi(x-t)dt\right] \qquad \text{(II.18)}$$

where K_+ and K_- denote the cones

$$K_+=\left\{t\in R^n : P(t)\geq 0\right\},$$
$$K_-=\left\{t\in R^n : P(t)\leq 0\right\}.$$

The integral in (II.18) converges if $\alpha>n-2\gamma$ and in the case $\alpha\leq n-2\gamma$ it admits an analytical continuation respecto to α (cf. [10]).

III. THE GENERALIZED BESSEL-RIESZ DERIVATIVE

To obtain an inverse operator of W^{α}, which is indicated by $\left(W^{\alpha}\right)^{-1}$, such that $f=W^{\alpha}\varphi$ it results that $\varphi=\left(W^{\alpha}\right)^{-1}f$, we introduce an operator $\left(W^{\alpha}\right)^{-1}$ that is a linear combination of hypersingular integrals of orders $\alpha-2\gamma$, $\gamma=0,1,\ldots,\left[\frac{\alpha}{2}\right]$ plus an integral operator

$$\left(W^{\alpha}\right)^{-1}(f)=\sum_{\gamma=0}^{\left[\frac{\alpha}{2}\right]}\binom{\frac{\alpha}{2}}{\gamma}\frac{m^{2\gamma}}{d_{n,l}(\alpha-2\gamma)}T_{l,e}^{\alpha-2\gamma}f+\sum_{\gamma=\left[\frac{\alpha}{2}\right]+1}^{\infty}\binom{\frac{\alpha}{2}}{\gamma}m^{2\gamma}\frac{R_{-\alpha+2\gamma}}{H(-\alpha+2\gamma)}*f \qquad \text{(III.1)}$$

where

$$\left(T_{l,\varepsilon,\gamma}^{\alpha-2\gamma}f\right)(x)=\int_{R^n}\left(P+i\varepsilon|t|^2\right)^{-\frac{n+\alpha-2\gamma}{2}}\left\{\left(\Delta_t^l f\right)\right\}dt\,; \qquad \text{(III.2)}$$

where $\left(\Delta_t^l f\right)(x)=\sum_{k=0}^{\infty}\binom{l}{k}(-1)^k f(x-kt)$ is the difference of order l of the function f at the point x with interval t. The operator $T_{l,\varepsilon,\gamma}^{\alpha-2\gamma}$ shall be defined as "the hypersingular integral in differences" and it is a causal analogue of the integral definied by Samko (cf. [10]) for the elliptic case, and by Rubin ([8]) for the Bessel potentials and by us (cf. [1]) for causal Bessel potentials and the same for causal Riesz potentials (cf. [2] and [3]). And its Fourier transform is

$$\Im\left[T_{l,\varepsilon,\gamma}^{\alpha-2\gamma}f\right](\xi)=d_{n,l}(\alpha-2\gamma)\left(Q\mp i\varepsilon|\xi|^2\right)^{\frac{\alpha-2\gamma}{2}}\Im[f](\xi) \qquad \text{(III.3)}$$

where the constant $d_{n,l}(\alpha-2\gamma)$ is given by

$$d_{n,l}(\alpha-2\gamma)=$$

$$=\begin{cases}\dfrac{\pi^{\frac{n}{2}+1}e^{i\frac{\pi}{2}q}A_l(\alpha-2\gamma)}{2^{\alpha-2\gamma}\Gamma\left(1+\frac{\alpha-2\gamma}{2}\right)\Gamma\left(\frac{n+\alpha-2\gamma}{2}\right)\text{sen}\,\frac{\pi}{2}(\alpha-2\gamma)} & \text{if } \alpha-2\gamma\neq 2,4,6...\\ \dfrac{(-1)^{\alpha-2\gamma}\pi^{\frac{n}{2}}2^{1-(\alpha-2\gamma)}e^{i\frac{\pi}{2}q}}{\Gamma\left(1+\frac{\alpha-2\gamma}{2}\right)\Gamma\left(\frac{n+\alpha-2\gamma}{2}\right)}\dfrac{d}{d\alpha}A_l(\alpha-2\gamma) & \text{if } \alpha-2\gamma=2.4.6...\end{cases} \qquad \text{(III.4)}$$

This operator is such that it Fourier transform is.

$$\Im\left[\left(W^{\alpha}\right)^{-1}(f)\right]=\sum_{\gamma=0}^{\left[\frac{\alpha}{2}\right]}\binom{\frac{\alpha}{2}}{\gamma}m^{2\gamma}(Q-i0)^{\frac{n}{2}-\gamma}\Im[f]+$$

$$+\sum_{\gamma=\left[\frac{\alpha}{2}\right]+1}^{\infty}\binom{\frac{n}{2}}{\gamma}m^{2\gamma}(Q-i0)^{\frac{\alpha}{2}-\gamma}\Im[f]=\sum_{\gamma=0}^{\infty}\binom{\frac{n}{2}}{\gamma}m^{2\gamma}(Q-i0)^{\frac{\alpha}{2}-\gamma}\Im[f]$$

and taking into account that

$$\Im[W_{\alpha}(P\pm i0,m,n)]=\sum_{\gamma=0}^{\infty}\binom{-\frac{n}{2}}{\gamma}m^{2\gamma}(Q\mp i0)^{-\frac{\alpha+2\gamma}{2}} \qquad \text{(III.5)}$$

it result

$$\Im\left[\left(W^{\alpha}\right)^{-1}(f)\right]=\sum_{\gamma=0}^{\infty}\binom{\frac{a}{2}}{\gamma}m^{2\gamma}(Q-i0)^{\frac{\alpha}{2}-\gamma}=\Im[W_{-\alpha}*f]$$

Analogously to the Riesz derivative and causal Riesz derivative (cf. [9], [3] and [2]) and the causal Bessel derivative (cf. [1]) we define the generalized Bessel-Riesz derivative of order α of a function $f\in S$ when $\alpha\neq 1,3,5,\ldots$ by

$$\Im\left[D^{\alpha}f\right](\xi)=\sum_{\gamma=0}^{\infty}\binom{\frac{a}{2}}{\gamma}m^{2\gamma}(Q\mp i0)^{\frac{\alpha-2\gamma}{2}}\Im[f](\xi) \qquad \text{(III.6)}$$

IV. Inversion of Bessel-Riesz potentials defined on $S'(R^n)$.

In order to extend the inversion to Bessel-Riesz potentials defined on temperate distributions we need the relation between the derivative of certain order β and the Bessel-Riesz potential of order α of a function φ belonging to the space *S*. Let the operator $D^{\beta}W^{\alpha}\varphi$. To obtain an expression of this last operation we start by evaluate its Fourier transform.

$$\Im\left[D^{\beta}W^{\alpha}\varphi\right]=\sum_{\gamma\geq 0}\binom{\frac{\beta}{2}}{\gamma}m^{2\gamma}(Q-i0)^{\frac{\beta-2\gamma}{2}}\Im\left[W^{\alpha}\varphi\right]=$$

$$=\sum_{\gamma\geq 0}\binom{\frac{\beta}{2}}{\gamma}m^{2\gamma}(Q-i0)^{\frac{\beta-2\gamma}{2}}\cdot\sum_{n\geq 0}\binom{-\frac{\alpha}{2}}{\gamma}m^{2n}(Q\mp i0)^{-\frac{\alpha+2n}{2}}\Im[\varphi]=$$

$$=\sum_{\gamma\geq 0}\sum_{j=0}\binom{\frac{\beta}{2}}{j}\binom{-\frac{\alpha}{2}}{\gamma-j}m^{2\gamma}(Q-i0)^{\frac{\beta-\alpha-2\gamma}{2}}\Im[\varphi]=$$

$$=\sum_{\gamma\geq 0}\binom{\frac{\beta-\alpha}{2}}{\gamma}m^{2\gamma}(Q-i0)^{\frac{\beta-\alpha-2\gamma}{2}}\Im[\varphi]=$$

From (III.5) making the change $\alpha\to\alpha-\beta$, we obtain

$$\Im\left[W_{\alpha-\beta}*\varphi\right]=\Im\left[W^{\alpha-\beta}\varphi\right]$$

And by the uniqueness of the Fourier transform

$$D^{\beta}W^{\alpha}\varphi=W^{\alpha-\beta}\varphi$$

Thus, we have proved the following:

Theorem 2. Let α and β be real positive numbers, $\beta\leq\alpha$. Then is valid the following result

$$D^{\beta}W^{\alpha}\varphi=W^{\alpha-\beta}\varphi$$

Corollary: As a particular case when $\alpha=\beta$, $D^{\beta}W^{\alpha}\varphi=\varphi$.

In fact: From the last formulae, putting $\beta=\alpha$

$$D^{\alpha}W^{\alpha}\varphi=W^{\alpha-\alpha}\varphi=W^{0}\varphi=\delta*\varphi=\varphi.$$

Now we can extend the Bessel-Riesz operator to temperate distributions.

Definition 3. Let T be a distribution belonging to S', and $\alpha>0$. Then Bessel-Riesz potential $W^{\alpha}T$ is definied by the relation:

$$\left(W^{\alpha}T,\varphi\right)=\left(T,W^{\alpha}\varphi\right). \qquad \text{(IV.1)}$$

It is clear that (IV.1) defines a functional in S'.

For temperates distributions the following result holds.

Theorem 4. Let T_1 and T_2 be temperate distributions and $\alpha>0$. Then the two following assertions are equivalent

1. $T_1=W^{\alpha}T_2$, and
2. $T_2=\lim_{\varepsilon\to 0}D^{\alpha}_{\varepsilon}T_1$

Proof. We begin by proving $1)\Rightarrow 2)$.

We have

$$\lim_{\varepsilon \to 0}\left(D_\varepsilon^\alpha T_1,\varphi\right)=\lim_{\varepsilon \to 0}\left(T_1,D_\varepsilon^\alpha\varphi\right)=\lim_{\varepsilon \to 0}\left(W^\alpha T_2,D_\varepsilon^\alpha\varphi\right)=\lim_{\varepsilon \to 0}\left(T_2,W^\alpha D_\varepsilon^\alpha\varphi\right)\overset{(1)}{=}\left(T_2,\varphi\right) \quad \text{(IV.2)}$$

The identity (1) results from Corollary of Theorem 2.

Now we shall prove 2) $\Rightarrow$ 1).

If $T_2=\lim_{\varepsilon \to 0}D_\varepsilon^\alpha T_1$, we have

$$\left(W^\alpha T_2,\varphi\right)=\left(T_2,W^\alpha\varphi\right)=\lim_{\varepsilon \to 0}\left(D_\varepsilon^\alpha T_1,W^\alpha\varphi\right)=\lim_{\varepsilon \to 0}\left(T_1,D_\varepsilon^\alpha W^\alpha\varphi\right)=\left(T_1,\varphi\right) \quad \text{(IV.3)}$$

From (IV.2) and (IV.3) the theorem follows.

V. THE INVERSE OPERATOR $\left(W^\alpha\right)^{-1}$, FOR $\alpha=2k$, $\kappa=1,2,\ldots$ AS LINEAR COMBINATION OF CAUSAL RIESZ DERIVATIVES

We begin by consider the binomial expansion of the distribution

$$\left(m^2+P\pm i0\right)^k=\sum_{j=0}^{k}\binom{k}{j}\left(m^2\right)^{k-j}\left(P\pm i0\right)^j \quad \text{(V.1)}$$

and remembering that

$$\left(m^2+P\pm i0\right)^k=\left(m^2+P-i0\right)^k=\left(m^2+P\right)^k, \text{ and}$$
$$\left(P\pm i0\right)^k=\left(P-i0\right)^k=\left(P\right)^k \text{ (cf. [?]),} \quad \text{(V.2)}$$

result that

$$\left(m^2+P\right)^k=\sum_{j=0}^{k}\binom{k}{j}\left(m^2\right)^{k-j}P^j \quad \text{(V.3)}$$

Taking into account the inversion theorem for Bessel-Riesz potentials we have

$$\Im\left[\left(W^{2k}\right)^{-1}f\right]=\Im\left[D^{2k}f\right]=\Im\left[(m^2+\square)^k f\right]=\left(m^2+Q\right)^k\Im[f] \quad \text{(V.4)}$$

Putting (V.4) in (V.3)

$$\Im\left[\left(W^{2k}\right)^{-1}f\right]=\sum_{j=0}^{k}\binom{k}{j}\left(m^2\right)^{k-j}\left(Q-i0\right)^j\Im[f] \quad \text{(V.5)}$$

The Fourier transform of the causal Riesz derivative is given by

$$\Im\left[D^\alpha f\right]=\left(Q-i0\right)^{\frac{\alpha}{2}}\Im[f] \text{ (cf. [2])} \quad \text{(V.6)}$$

then

$$\Im\left[\left(W^{2k}\right)^{-1}f\right]=\sum_{j=0}^{k}\binom{k}{j}\left(m^2\right)^{k-j}\Im\left[D^{2j}f\right] \quad \text{(V.7)}$$

and it results

$$\left(W^{2k}\right)^{-1} f = \sum_{j=0}^{k} \binom{k}{j} \left(m^2\right)^{k-j} \Im\left[D^{2j} f\right] \qquad \text{(V.8)}$$

Moreover, taking into account that for causal Riesz derivative of order $2j$, j a non negative integer we have

$$\Im\left[D^{2j} f\right] = \Im\,[\Box^{j} f] \ \text{(cf. [2])} \qquad \text{(V.9)}$$

where € denotes the ultrahyperbolic differential operator

$$€ = \frac{\partial^2}{\partial t_1^2} + \cdots + \frac{\partial^2}{\partial t_p^2} - \frac{\partial^2}{\partial t_{p+1}^2} - \cdots - \frac{\partial^2}{\partial t_{p+q}^2}$$

Then from (V.8) we arrive at

$$\left(W^{2k}\right)^{-1} = \sum_{j=0}^{k} \binom{k}{j} \left(m^2\right)^{k-j} \Box^{j} f \qquad \text{(V.10)}$$

This last formula is analogue to the following due to Samko obtained for the elliptic Riesz potential (cf. [9])

$$\left(B^{\alpha}\right)^{-1} = \sum_{j=0}^{\frac{\alpha}{2}} \binom{\frac{\alpha}{2}}{j} (\Delta)^{j} f \qquad \text{(V.11)}$$

where $\left(B^{\alpha}\right)^{-1}$ is the inverse operator of the Bessel operator of order α and Δ denote the Laplacian operator.

VI. RELATIONS BETWEEN THE BESSEL-RIESZ OPERATORS AND THE KLEIN-GORDON OPERATOR

If $K^{l} = \{\Box + m^2\}^{l}$ designates the ultrahyperbolic Klein-Gordon differential operator iterated l times, it was proved (cf. [14]) that $W_{2l}\left(P \pm io, m, n\right)$ is an elementary solution, i.e.

$$\{\Box + m^2\}^{l} W_{2l}\left(P \pm io, m, n\right) = \delta \qquad \text{(VI.1)}$$

From this fact it may be proved the following

Theorem 5. Let α be a real number, $\alpha \geq 2l$; $l = 1,2,\ldots$ Let K^{l} be the Klein-Gordon operator iterated l times and let $W^{\alpha}\varphi$ be the Bessel-Riesz operator of order α and φ; then

$$K^{l}\left\{W^{\alpha}\varphi\right\} = W^{\alpha-2l}\varphi.$$

Proof. By definition (II.13) we have

$$W^{\alpha-2l}\varphi = W_{\alpha-2l}\left(P \pm io, m, n\right) * \varphi \qquad \text{(VI.2)}$$

From (II.13), (IV.1) we obtain

$$W^{\alpha-2l}\varphi = W_{\alpha-2l} * \varphi = W_{\alpha} * W_{-2l} * \varphi = W_{\alpha} * K^{l}\varphi = W_{\alpha}\left\{K^{l}\varphi\right\} \qquad \text{(VI.3)}$$

and analogously

$$W^{\alpha-2l}\varphi = K^l\left\{W^\alpha\varphi\right\} \qquad \text{(VI.4)}$$

Then, from (VI.3) and (VI.4) it results

$$K^l\left\{W^\alpha\varphi\right\} = W^{\alpha-2l}\varphi \qquad \text{(VI.5)}$$

Theorem 6. The same hypothesis of Theorem 5. Then

$$W^\alpha K^l\varphi = W^{\alpha-2l}\varphi$$

Proof. The proof is analogue to the proof of Theorem 5.

In this paragraph we obtain an expression that will be consider a negative fractional power of the Klein-Gordon operator. The fractional power of a differential operator here is interpreted in the same way that Samko (cf. [10])

The Klein-Gordon operator is given by

$$(\Box + m^2) = \left\{\frac{\partial^2}{\partial t_1^2} + \cdots + \frac{\partial^2}{\partial t_p^2} - \frac{\partial^2}{\partial t_{p+1}^2} - \cdots - \frac{\partial^2}{\partial t_{p+q}^2} + m^2\right\}$$

From the fact that the application of the operator is reduce by Fourier transform to the following form

$$\Im\,[-(\Box + m^2)\varphi] = \left(m^2 + P(t)\right)\Im[\varphi] \qquad \text{(VI.6)}$$

i.e.: it is reduced to the multiplication by $m^2 + P$, we introduce the fractional power of the Klein-Gordon operator as an operator which are defined in terms of Fourier transforms by means of multiplication by a fractional power of the $\left(m^2 + P\right)$ generalized function.

From (VI.6) and (II.4) we may introduce an fractional power of the Klein-Gordon operator as

$$[-(\Box + m^2)]^\alpha\varphi = \Im^{-1}\left[\left(m^2 + Q \mp io\right)^\alpha\right]\Im[\varphi]$$

Taking into account that the fractional power of the D'Alembertain is given by

$$[-\Box]^\alpha\varphi = \Im^{-1}\left[(Q \mp io)^\alpha\right]\Im[\varphi] \text{ (cf. [10])}$$

the formulae (II.17) may be written

$$\Im\left[W^\alpha\varphi\right] = \sum_{\gamma=0}^{\infty}\binom{-\frac{\alpha}{2}}{\gamma} m^{2\gamma}(Q \mp io)^{-\frac{\alpha+2\gamma}{2}}\Im[\varphi]$$

$$= \sum_{\gamma=0}^{\infty}\binom{-\frac{\alpha}{2}}{\gamma} m^{2\gamma}\Im\,[\Box^{-\frac{\alpha}{2}+\gamma}\varphi\,]$$

$$= \Im\,[(\Box + m^2)^{-\frac{\alpha}{2}}\varphi\,] \qquad \text{(VI.7.)}$$

Then by the uniqueness of the Fourier transform we get

$$W^{\alpha}\varphi = (\Box + m^2)^{-\frac{\alpha}{2}}\varphi \qquad \text{(VI.8)}$$

in S' sense.

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